

GROUND STATES FOR SEMI-RELATIVISTIC SCHRÖDINGER-POISSON-SLATER ENERGIES

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ABSTRACT. We prove compactness (up to translation) of minimizing sequences to:

$$I_p^{\alpha,\beta}(\rho) = \inf_{\substack{u \in H^{\frac{1}{2}}(\mathbb{R}^3) \\ \int_{\mathbb{R}^3} |u|^2 dx = \rho}} \frac{1}{2} \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \alpha \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \beta \int_{\mathbb{R}^3} |u|^p dx$$

where $2 < p \leq \frac{8}{3}$, $\alpha, \beta > 0$ and $\rho > 0$ is small enough. In the case $p = \frac{8}{3}$ we show that similar compactness properties fail provided that in the energy above we replace the inhomogeneous Sobolev norm $\|u\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2$ by the homogeneous one $\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$. We also provide a characterization of the parameters $\alpha, \beta > 0$ in such a way that $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) > -\infty$ for every $\rho > 0$.

In this paper we analyse compactness properties of minimizing sequences to the following minimization problems:

$$(0.1) \quad I_p^{\alpha,\beta}(\rho) = \inf_{u \in S(\rho)} \mathcal{E}_p^{\alpha,\beta}(u)$$

where

$$(0.2) \quad \mathcal{E}_p^{\alpha,\beta}(u) = \frac{1}{2} \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \alpha \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \beta \int_{\mathbb{R}^3} |u|^p dx,$$

$$\alpha, \beta > 0, 2 < p \leq \frac{8}{3}$$

$$(0.3) \quad S(\rho) = \left\{ u \in H^{\frac{1}{2}}(\mathbb{R}^3) \text{ s.t. } \int_{\mathbb{R}^3} |u|^2 dx = \rho \right\}$$

and $H^s(\mathbb{R}^3)$ denotes for general $s \in \mathbb{R}$ the usual Sobolev spaces endowed with the norm:

$$\|u\|_{H^s(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$$

with $\hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x \xi} u(x) dx$.

By using the well-known property $\|w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2 \leq \|w\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2$, where equality occurs if and only if there exists $\theta \in \mathbb{R}$ such that $e^{i\theta} w$ is real-valued (see for instance [9]), one can deduce that if $v(x)$ is a minimizer for (0.1) then there exists $\theta \in \mathbb{R}$

such that $e^{i\theta}v$ is real-valued. In particular any minimizer v to (0.1) solves the following equation:

$$(0.4) \quad \sqrt{1-\Delta} v + 4\alpha(|x|^{-1} * |v|^2)v - \beta p|v|^{p-2}v = \omega v \quad \text{in } \mathbb{R}^3$$

for a suitable Lagrange multiplier $\omega \in \mathbb{R}$. Moreover the corresponding time-dependent function

$$(0.5) \quad \psi(x, t) = e^{-i\omega t}v(x)$$

is a solution of the time-dependent Nonlinear Schrödinger Equation

$$(0.6) \quad i\psi_t = \sqrt{1-\Delta} \psi + 4\alpha(|x|^{-1} * |\psi|^2)\psi - \beta p|\psi|^{p-2}\psi.$$

We recall that solutions of the type (0.5) are known in the literature as solitary waves. Equation (0.6) corresponds, for the value $p = \frac{8}{3}$, to the semi-relativistic Schrödinger-Poisson-Slater equation which arises in the approximation of the Hartree-Fock model for N particle interacting with each other via the Coulomb law (see [14]).

The existence of ground states for the semirelativistic Schrödinger Equation (0.4) with $\beta = 0$ and $\alpha < 0$ it has been proved in [6], [8], [10], [11]. The existence of ground states for non relativistic Schrödinger-Poisson-Slater equation (i.e. (0.4) where $\sqrt{1-\Delta}$ is replaced by the Laplacian operator $-\Delta$) has been proved in [13] in the special case $p = \frac{8}{3}$ and $\alpha, \beta > 0$, and extended in [1] and [2] respectively in the cases $3 < p < \frac{10}{3}$ and $2 < p < 3$. Finally we quote [5] where it is studied the non relativistic Schrödinger-Poisson-Slater equation with the nonlinearity $|u|^{\frac{10}{3}} - |u|^{\frac{8}{3}}$.

Let us recall that the interest in looking at the minimization problem (0.1) is twofold: on one hand it provides the existence of solitary waves; on the other hand (following the very general argument in [5]) the solitary waves associated to minimizers are expected to be orbitally stable for the dynamic associated to (0.6). Indeed the notion of orbital stability can be given in a rigorous way provided that the Cauchy problem associated to (0.6) is globally well-posed. As far as we know this evolutionary problem has not been studied in the literature and we plan to pursue it in the next future. In this context we quote the paper [12] where it is studied the following Cauchy problem:

$$(0.7) \quad i\psi_t = \sqrt{1-\Delta}\psi - (|x|^{-1} * |\psi|^2)\psi.$$

In this case the main advantage is the smoothing effect associated to the Hartree nonlinearity which allows to solve the Cauchy problem by using the classical energy estimates. On the contrary the nonlinearity in (0.6) does not enjoy the same smoothness and it makes more complicated the analysis of the corresponding Cauchy problem.

Nevertheless the minimization problem (0.1) has its own interest and can be handled with technology which is completely independent on the one needed for the analysis of the Cauchy problem.

Recall that a general strategy to attack constrained minimization problems is the celebrated concentration-compactness principle of P.L. Lions, see [12]. The main point is that in general if u_n is a minimizing sequence for (0.1) then up to translations two possible bad scenarios can occur (that can be shortly summarized as follows):

- (vanishing) $u_n \rightharpoonup 0$;
- (dichotomy) $u_n \rightharpoonup \bar{u} \neq 0$ and $0 < \|\bar{u}\|_2 < \rho$.

Typically the vanishing can be excluded by proving that any minimizing sequence weakly converges, up to translation, to a function \bar{u} different from zero (in turn it can be accomplished in general by a suitable localized Gagliardo-Nirenberg inequality in conjunction with the Rellich compactness theorem).

Concerning the dichotomy the classical way to rule out it is by proving the following strong subadditivity inequality

$$(0.8) \quad I_p^{\alpha,\beta}(\rho) < I_p^{\alpha,\beta}(\mu) + I_p^{\alpha,\beta}(\rho - \mu) \quad \forall 0 < \mu < \rho.$$

Although the following weak version of (0.8)

$$(0.9) \quad I_p^{\alpha,\beta}(\rho) \leq I_p^{\alpha,\beta}(\mu) + I_p^{\alpha,\beta}(\rho - \mu) \quad \text{for all } 0 < \mu < \rho.$$

can be easily proved, in general the proof of (0.8) requires some extra arguments which heavily depend on the structure of the functional we are looking at.

In our concrete situation (i.e. (0.1)) the main difficulties to avoid vanishing and dichotomy are related to the following facts:

- the functional $\mathcal{E}_p^{\alpha,\beta}(u)$ involves three terms with different degrees of homogeneity;
- the quadratic term in $\mathcal{E}_p^{\alpha,\beta}(u)$ is a norm which is not rescaling invariant and nonlocal.

Next we state our first result.

Theorem 0.1. *For every $p \in (2, \frac{8}{3})$, $\alpha, \beta > 0$, $\rho > 0$ we have $I_p^{\alpha,\beta}(\rho) > -\infty$. Moreover there exists $\rho_1 = \rho_1(p, \alpha, \beta) > 0$ such that for every $0 < \rho < \rho_1$ and for every sequence u_n which satisfy:*

$$u_n \in S(\rho) \text{ and } \mathcal{E}_p^{\alpha,\beta}(u_n) \rightarrow I_p^{\alpha,\beta}(\rho)$$

there exists, up to subsequence, $\tau_n \in \mathbb{R}^3$ such that

$$u_n(\cdot + \tau_n) \text{ has a strong limit in } H^{\frac{1}{2}}(\mathbb{R}^3).$$

In particular the set of minimizers for $I_p^{\alpha,\beta}(\rho)$ is not empty (for ρ small) and any minimizer v satisfies:

$$(0.10) \quad \frac{1}{2}\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2 - \frac{1}{2}\|v\|_{H^{-\frac{1}{2}}(\mathbb{R}^3)}^2 + \alpha \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \left(\frac{3p-6}{2} \right) \|v\|_{L^p(\mathbb{R}^3)}^p = 0.$$

Next we focus in the limit case $p = \frac{8}{3}$, which is the most interesting case from a physical point of view. The first result concerns the characterization of the values $\alpha, \beta > 0$ such that $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) > -\infty$ for every $\rho > 0$.

We need to introduce the constant S defined as follows:

$$S = \inf\{C \in (0, \infty] \text{ s.t. } C \text{ satisfies (0.11)}\}$$

$$(0.11) \quad \|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)} \leq C \|\varphi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{\frac{1}{2}} \left(\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy \right)^{\frac{1}{8}} \\ \forall \varphi \in C_0^\infty(\mathbb{R}^3)$$

where $\|\varphi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\xi| |\hat{\varphi}(\xi)|^2 d\xi$. In the appendix we prove that the estimate (0.11) is true and hence $S < \infty$ is its best constant.

Theorem 0.2. *Let $\alpha, \beta > 0$ be fixed. Then the following facts are equivalent:*

- $\exists \rho > 0$ s.t. $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) = -\infty$;
- $\left(\frac{27\alpha}{\beta^3} \right)^{\frac{1}{8}} < \sqrt{2}S$.

Next we state an analogue version of theorem 0.1 in the case $p = \frac{8}{3}$.

Theorem 0.3. *For every $\alpha, \beta > 0$ there exists $\bar{\rho} = \bar{\rho}(\alpha, \beta) > 0$ such that $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) \geq 0$ for every $0 < \rho < \bar{\rho}$. Moreover for every sequence u_n which satisfy:*

$$u_n \in S(\rho) \text{ and } \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(u_n) \rightarrow I_{\frac{8}{3}}^{\alpha,\beta}(\rho), \text{ with } 0 < \rho < \bar{\rho}$$

there exists, up to subsequence, $\tau_n \in \mathbb{R}^3$ such that

$$u_n(\cdot + \tau_n) \text{ has a strong limit in } H^{\frac{1}{2}}(\mathbb{R}^3).$$

In particular the set of minimizers for $I_{\frac{8}{3}}^{\alpha,\beta}(\rho)$ is not empty (for ρ small) and any minimizer v satisfies the following identity:

$$(0.12) \quad \frac{1}{2}\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^2 - \frac{1}{2}\|v\|_{H^{-\frac{1}{2}}(\mathbb{R}^3)}^2 + \alpha \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \|v\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^{\frac{8}{3}} = 0.$$

In our opinion theorem 0.3 is quite surprising in view of the next nonexistence result. First we introduce the following minimization problems

$$\tilde{I}_p^{\alpha,\beta}(\rho) = \inf_{u \in S(\rho)} \tilde{\mathcal{E}}_p^{\alpha,\beta}(u)$$

where

$$(0.13) \quad \tilde{\mathcal{E}}_p^{\alpha,\beta}(u) = \frac{1}{2} \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \alpha \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \beta \int_{\mathbb{R}^3} |u|^p dx.$$

and

$$\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\xi| |\hat{u}(\xi)|^2 d\xi$$

(here $\hat{u}(\xi)$ denotes the Fourier transform of u and $S(\rho)$ is defined in (0.3)). Notice that the unique difference between $\mathcal{E}_p^{\alpha,\beta}$ and $\tilde{\mathcal{E}}_p^{\alpha,\beta}$ concerns the quadratic part which in the second case is an homogeneous norm, while in the first case is the inhomogeneous one.

Theorem 0.4. *For every $\alpha, \beta > 0$ there exists $\bar{\rho} = \bar{\rho}(\alpha, \beta) > 0$ such that:*

- $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) > -\infty \quad \forall \rho \in (0, \bar{\rho})$;
- $\forall \rho \in (0, \bar{\rho})$ and $\forall v \in S(\rho)$ we have $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(v) > \tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho)$
(i.e. there are not minimizers for $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho)$ with ρ small).

Remark 0.1. Despite to theorem 0.4 one can prove a version of theorem 0.1 in case that $\mathcal{E}_p^{\alpha,\beta}$ is replaced by the modified energy $\tilde{\mathcal{E}}_p^{\alpha,\beta}$ with $p \in (2, \frac{8}{3})$.

In the sequel we shall use the following

Notation.

$$\begin{aligned} \|u\|_q^q &= \int_{\mathbb{R}^3} |u|^q dx \quad \forall 1 \leq q < \infty; \\ \int f(x) dx &= \int_{\mathbb{R}^3} f(x) dx \quad \text{and} \quad \int \int F(x, y) dx dy = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} F(x, y) dx dy; \\ \|u\|_{H^s}^2 &= \int \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi \quad \forall s \in \mathbb{R} \quad \text{where} \\ \langle \xi \rangle &= \sqrt{1 + |\xi|^2} \quad \text{and} \quad \hat{u}(\xi) = \int e^{-2\pi i x \xi} u(x) dx; \\ \forall s \in \mathbb{R}, R > 0 \quad \text{and} \quad u \in H^s \quad \text{we denote} \\ B_{H^s}(u, R) &= \{w \in H^s \text{ s.t. } \|w - u\|_{H^s} < R\}; \\ \|u\|_{H^s}^2 &= \int |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \quad \forall s \in \mathbb{R}; \\ \forall \rho > 0 \quad S(\rho) &\text{ is the set defined in (0.3).} \end{aligned}$$

The paper is organized as follows: in section 1 we prove functional identities satisfied by minimizers of suitable minimization problems (this will be the key ingredient to kill dichotomy); section 2 is devoted to the analysis of qualitative properties of the real function $\rho \rightarrow I_p^{\alpha,\beta}(\rho)$; in section 3 we present a suitably adapted version in $H^{\frac{1}{2}}$ of the concentration compactness developed by [12]; in sections 4 and 5 we prove theorems 0.1 and 0.3 respectively; sections 6 and 7 are devoted to the proof of theorems 0.2 and 0.4; in the appendix we prove inequality (0.11).

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1. SOME FUNCTIONAL IDENTITIES FOR MINIMIZERS

The following result will be the key point to avoid dichotomy for minimizing sequences.

Proposition 1.1. *Let $\alpha, \beta > 0$, $p \in (2, \frac{8}{3}]$ be fixed. Let $v \in H^{\frac{1}{2}}$ be a local minimizer to*

$$J_p^{\alpha,\beta} : H^{\frac{1}{2}} \setminus \{0\} \ni u \rightarrow \mathcal{E}_p^{\alpha,\beta}(u) / \|u\|_2^2 \in \mathbb{R}$$

(i.e. $\exists \epsilon > 0$ s.t. $J_p^{\alpha,\beta}(u) \geq J_p^{\alpha,\beta}(v) \forall u \in B_{H^{\frac{1}{2}}}(v, \epsilon)$)

then

$$(1.1) \quad 2\alpha \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta(p-2) \|v\|_p^p = 0.$$

Proof. Let us introduce for every $\theta \in (0, \infty)$ the rescaled function

$$v_\theta(x) = \theta v(x).$$

Then

$$\|v_\theta - v\|_{H^{\frac{1}{2}}} \rightarrow 0 \text{ as } \theta \rightarrow 1$$

and hence by assumption the function

$$f_p^{\alpha,\beta} : (0, \infty) \ni \theta \rightarrow J_p^{\alpha,\beta}(v_\theta) \in \mathbb{R}$$

has a local minimum in $\theta = 1$. By explicit computations we get

$$f_p^{\alpha,\beta}(\theta) = \frac{\|v\|_{H^{\frac{1}{2}}}^2}{2\|v\|_2^2} + \alpha \frac{\theta^2}{\|v\|_2^2} \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \theta^{p-2} \frac{\|v\|_p^p}{\|v\|_2^2}.$$

In particular $f_p^{\alpha,\beta}$ is differentiable at $\theta = 1$ and since it has a minimum in $\theta = 1$ necessarily $\frac{d}{d\theta} f_p^{\alpha,\beta}[1] = 0$. This last condition is equivalent to (1.1). \square

Next we prove that if $v \in S(\rho)$ is a minimizer for $I_p^{\alpha,\beta}(\rho)$ then the identities (0.10) and (0.12) are satisfied.

Proposition 1.2. *Assume $v \in S(\rho)$ is a minimizer for $I_p^{\alpha,\beta}(\rho)$ with $p \in (2, \frac{8}{3}]$ and $\alpha, \beta > 0$. Then v satisfies the following identity:*

$$(1.2) \quad \frac{1}{2} \|v\|_{H^{\frac{1}{2}}}^2 - \frac{1}{2} \|v\|_{H^{-\frac{1}{2}}}^2 + \alpha \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \left(\frac{3p-6}{2} \right) \|v\|_p^p = 0.$$

Proof. Notice that

$$v_\theta = \theta^{\frac{3}{2}} v(\theta x) \in S(\rho).$$

Since v is a minimizer for $I_p^{\alpha,\beta}(\rho)$ the function

$$g_p^{\alpha,\beta} : (0, \infty) \ni \theta \rightarrow \mathcal{E}_p^{\alpha,\beta}(v_\theta)$$

has a minimum at $\theta = 1$. Notice that by explicit computation

$$g_p^{\alpha,\beta}(\theta) = \frac{1}{2} \int \sqrt{1 + \theta^2 |\xi|^2} |\hat{v}|^2 d\xi + \alpha \theta \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \theta^{\frac{3}{2}p-3} \|v\|_p^p$$

which implies regularity of $g_p^{\alpha,\beta}(\theta)$. In particular $\frac{d}{d\theta} g_p^{\alpha,\beta}[1] = 0$ and hence

$$\frac{1}{2} \int \frac{|\xi|^2}{\sqrt{1 + |\xi|^2}} |\hat{v}|^2 d\xi + \alpha \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dx dy - \beta \left(\frac{3p-6}{2} \right) \|v\|_p^p = 0$$

which is equivalent to (1.2). □

2. SOME PROPERTIES OF THE FUNCTION $\rho \rightarrow I_p^{\alpha,\beta}(\rho)$

In this section we analyse some qualitative properties of the function $\rho \rightarrow I_p^{\alpha,\beta}(\rho)$.

Proposition 2.1. *Let $\alpha, \beta > 0$, $\rho > 0$ and $p \in (2, \frac{8}{3})$ be fixed. Assume that $u_n \in S(\rho)$ is a minimizing sequence for $I_p^{\alpha,\beta}(\rho)$ then:*

$$\sup_n \|u_n\|_{H^{\frac{1}{2}}} < \infty.$$

In particular $I_p^{\alpha,\beta}(\rho) > -\infty$.

Proof. By using the Hölder inequality in conjunction with the embedding $H^{\frac{1}{2}} \subset L^3$ we deduce for n large enough the following estimate:

$$(2.1) \quad \begin{aligned} I_p^{\alpha,\beta}(\rho) + 1 &\geq \mathcal{E}_p^{\alpha,\beta}(u_n) \geq \frac{1}{2} \|u_n\|_{H^{\frac{1}{2}}}^2 - C \|u_n\|_2^{6-2p} \|u_n\|_3^{3p-6} \\ &\geq \frac{1}{2} \|u_n\|_{H^{\frac{1}{2}}}^2 - C \rho^{3-p} \|u_n\|_{H^{\frac{1}{2}}}^{3p-6} = h_{\rho,p}(\|u_n\|_{H^{\frac{1}{2}}}) \end{aligned}$$

where $h_{\rho,p}(t) = \frac{1}{2} t^2 - C \rho^{3-p} t^{3p-6}$. We can conclude since $\lim_{t \rightarrow \infty} h_{\rho,p}(t) = \infty$ provided that $p \in (2, \frac{8}{3})$. □

Proposition 2.2. *For every $\alpha, \beta > 0$ there exists $\rho_0 = \rho_0(\beta) > 0$ such that*

$$I_{\frac{8}{3}}^{\alpha, \beta}(\rho) \geq 0 \quad \forall 0 < \rho < \rho_0.$$

Moreover if $u_n \in S(\rho)$ is a minimizing sequence for $I_{\frac{8}{3}}^{\alpha, \beta}(\rho)$ with $0 < \rho < \rho_0$ then

$$\sup_n \|u_n\|_{H^{\frac{1}{2}}} < \infty.$$

Proof. It follows from the following estimate

$$\mathcal{E}_{\frac{8}{3}}^{\alpha, \beta}(\varphi) \geq \frac{1}{2} \|\varphi\|_{H^{\frac{1}{2}}}^2 - C \|\varphi\|_2^{\frac{2}{3}} \|\varphi\|_{H^{\frac{1}{2}}}^2$$

where we have used the Hölder inequality in conjunction with the Sobolev embedding $H^{\frac{1}{2}} \subset L^3$. □

Proposition 2.3. *For every $\alpha, \beta > 0$, $p \in (2, \frac{8}{3})$ the function*

$$(0, \infty) \ni \rho \rightarrow I_p^{\alpha, \beta}(\rho) \in \mathbb{R}$$

is continuous.

Proof. Assume it is not continuous, then there exists a sequence ρ_n and $\epsilon > 0$ such that $\lim_{n \rightarrow \infty} \rho_n = \bar{\rho} > 0$ and $|I_p^{\alpha, \beta}(\rho_n) - I_p^{\alpha, \beta}(\bar{\rho})| \geq \epsilon > 0$. In particular up to subsequence we can assume that either

$$(2.2) \quad I_p^{\alpha, \beta}(\rho_n) - I_p^{\alpha, \beta}(\bar{\rho}) \geq \epsilon$$

or

$$(2.3) \quad I_p^{\alpha, \beta}(\bar{\rho}) - I_p^{\alpha, \beta}(\rho_n) \geq \epsilon.$$

First we shall prove that (2.2) cannot occur. We fix $w \in H^{\frac{1}{2}}$ such that

$$(2.4) \quad w \in S(\bar{\rho}) \text{ and } \mathcal{E}_p^{\alpha, \beta}(w) - I_p^{\alpha, \beta}(\bar{\rho}) \leq \frac{\epsilon}{2}$$

and we introduce

$$w_n = \sqrt{\frac{\rho_n}{\bar{\rho}}} w.$$

Notice that

$$(2.5) \quad w_n \in S(\rho_n) \text{ and } \lim_{n \rightarrow \infty} \mathcal{E}_p^{\alpha, \beta}(w_n) = \mathcal{E}_p^{\alpha, \beta}(w).$$

By combining (2.5) with (2.4) we get the existence of $\bar{n} \in \mathbb{N}$ such that

$$(2.6) \quad I_p^{\alpha, \beta}(\rho_n) \leq \mathcal{E}_p^{\alpha, \beta}(w_n) \leq \mathcal{E}_p^{\alpha, \beta}(w) + \frac{\epsilon}{4} \leq I_p^{\alpha, \beta}(\bar{\rho}) + \frac{3}{4}\epsilon \quad \forall n > \bar{n}$$

which is in contradiction with (2.2).

In order to contradict (2.3) we argue as follows. Let $v_n \in H^{\frac{1}{2}}$ such that

$$(2.7) \quad v_n \in S(\rho_n) \text{ and } \mathcal{E}_p^{\alpha, \beta}(v_n) - I_p^{\alpha, \beta}(\rho_n) \leq \frac{\epsilon}{2}.$$

We state the following

Claim *We can choose a sequence v_n that satisfies (2.7) and moreover*

$$\sup_n \|v_n\|_{H^{\frac{1}{2}}} < \infty.$$

By assuming the claim it is easy to prove that

$$(2.8) \quad \lim_{n \rightarrow \infty} (\mathcal{E}_p^{\alpha, \beta}(v_n) - \mathcal{E}_p^{\alpha, \beta}(u_n)) = 0.$$

where

$$u_n = \sqrt{\frac{\bar{\rho}}{\rho_n}} v_n \in S(\bar{\rho}).$$

By combining (2.7) with (2.8) we get the existence of $\bar{n} \in \mathbb{N}$ such that

$$I_p^{\alpha, \beta}(\bar{\rho}) \leq \mathcal{E}_p^{\alpha, \beta}(u_n) \leq \mathcal{E}_p^{\alpha, \beta}(v_n) + \frac{\epsilon}{4} \leq I_p^{\alpha, \beta}(\rho_n) + \frac{3}{4}\epsilon \quad \forall n > \bar{n}$$

hence contradicting (2.3).

Next we shall prove the claim. Notice that if (2.3) is true then

$$K = \sup_n I_p^{\alpha, \beta}(\rho_n) < \infty$$

and hence by looking at the estimate (2.1) we deduce that v_n can be chosen in such a way that:

$$K + 1 \geq h_{\rho_n, p}(\|v_n\|_{H^{\frac{1}{2}}})$$

where $h_{\rho_n, p}(t) = \frac{1}{2}t^2 - C\rho_n^{3-p}t^{3p-6}$. It is now easy to deduce the claim since for every $M > 0$ there exists $R > 0$ such that

$$h_{\rho_n, p}(t) \geq M \quad \forall t \geq R \quad \forall n \in \mathbb{N}.$$

□

Proposition 2.4. *Let $\alpha, \beta > 0$ be fixed and $\bar{\rho} = \bar{\rho}(\alpha, \beta) > 0$ be such that $I_{\frac{8}{3}}^{\alpha, \beta}(\rho) > -\infty$ for $\rho \in (0, \bar{\rho})$. Then the function*

$$(0, \bar{\rho}) \ni \rho \rightarrow I_{\frac{8}{3}}^{\alpha, \beta}(\rho) \in \mathbb{R}$$

is continuous.

Proof. The same proof of proposition 2.3.

□

Proposition 2.5. *For every $\alpha, \beta > 0$ and $p \in (2, \frac{8}{3})$ we have:*

$$(2.9) \quad \exists \rho_1 = \rho_1(\alpha, \beta, p) > 0 \text{ s.t. } \frac{I_p^{\alpha, \beta}(\rho)}{\rho} < \frac{1}{2} \quad \forall 0 < \rho < \rho_1;$$

$$(2.10) \quad \lim_{\rho \rightarrow 0} \frac{I_p^{\alpha, \beta}(\rho)}{\rho} = \frac{1}{2}.$$

Moreover there exists $C = C(\alpha, \beta, p) > 0$ such that for any $\rho \in (0, 1)$ we have:

$$(2.11) \quad \text{if } v_\rho \in S(\rho), I_p^{\alpha, \beta}(v_\rho) = I_p^{\alpha, \beta}(\rho) \text{ then } \|v_\rho\|_{H^{\frac{1}{2}}} < C\sqrt{\rho}.$$

Proof.

Proof of (2.9).

We introduce the functional

$$(2.12) \quad \begin{aligned} \mathcal{F}_p^{\alpha, \beta}(u) &= \mathcal{E}_p^{\alpha, \beta}(u) - \frac{1}{2}\|u\|_2^2 = \mathcal{E}_p^{\alpha, \beta}(u) - \frac{1}{2}\|\hat{u}\|_2^2 \\ &= \frac{1}{2} \int \frac{|\xi|^2}{1 + \langle \xi \rangle} |\hat{u}|^2 d\xi + \alpha \int \int \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dx dy - \beta \|u\|_p^p \end{aligned}$$

where we have used the Plancharel identity. Notice that (2.9) is equivalent to show that

$$(2.13) \quad \inf_{u \in S(\rho)} \mathcal{F}_p^{\alpha, \beta}(u) < 0 \quad \forall 0 < \rho < \rho_1$$

with ρ_1 small enough. In order to prove (2.13) we fix $\varphi \in C_0^\infty(\mathbb{R}^3)$ and we introduce $\varphi_\theta = \theta^\gamma \varphi(\theta x)$ where γ will be choosen later. Notice that by looking at the expression of $\mathcal{F}_p^{\alpha, \beta}$ in (2.12) we get

$$\begin{aligned} \inf_{u \in S(\theta^{2\gamma-3})} \mathcal{F}_p^{\alpha, \beta}(u) &\leq \mathcal{F}_p^{\alpha, \beta}(\varphi_\theta) \leq \frac{1}{2} \int |\xi|^2 |\hat{\varphi}_\theta|^2 d\xi \\ &\quad + \alpha \int \int \frac{|\varphi_\theta(x)|^2 |\varphi_\theta(y)|^2}{|x - y|} dx dy - \beta \|\varphi_\theta\|_p^p \\ &= \frac{1}{2} \theta^{2\gamma-1} \|\varphi\|_{H^1}^2 + \alpha \theta^{4\gamma-5} \int \int \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x - y|} dx dy - \beta \theta^{p\gamma-3} \|\varphi\|_p^p. \end{aligned}$$

Notice that the r.h.s. above is negative for $0 < \gamma < \bar{\gamma}$ provided that we can choose γ such that

$$2\gamma - 1 > 0, 4\gamma - 5 > 0, p\gamma - 3 > 0$$

$$p\gamma - 3 < 2\gamma - 1, p\gamma - 3 < 4\gamma - 5.$$

In fact the conditions above are satisfied for any $\gamma \in (\frac{2}{4-p}, \frac{2}{p-2})$ (notice that for every $2 < p < 3$ we have $\frac{2}{4-p} < \frac{2}{p-2}$).

Proof of (2.10).

Due to (2.9) it is sufficient to prove that

$$\liminf_{\rho \rightarrow 0} \frac{I_p^{\alpha, \beta}(\rho)}{\rho} \geq \frac{1}{2}.$$

For every $\rho > 0$ we fix a minimizing sequence $u_n \in S(\rho)$ for $I_p^{\alpha,\beta}(\rho)$ hence we have

$$\frac{I_p^{\alpha,\beta}(\rho)}{\rho} \geq \limsup_{n \rightarrow \infty} \left(\frac{1}{2} \rho^{-1} \|u_n\|_{H^{\frac{1}{2}}}^2 - \beta \rho^{-1} \|u_n\|_p^p \right).$$

Notice that $\limsup_{n \rightarrow \infty} \frac{1}{2} \rho^{-1} \|u_n\|_{H^{\frac{1}{2}}}^2 \geq \limsup_{n \rightarrow \infty} \frac{1}{2} \rho^{-1} \|u_n\|_{L^2}^2 = \frac{1}{2}$ hence it is sufficient to prove that

$$\limsup_{\rho \rightarrow 0} (\limsup_{n \rightarrow \infty} \rho^{-1} \|u_n\|_p^p) = 0.$$

This fact will follow by combining next claim with the usual Sobolev embedding.

Claim

$$(2.14) \quad \exists \bar{\rho} > 0, C > 0 \text{ s.t. } \limsup_{n \rightarrow \infty} \|u_n\|_{H^{\frac{1}{2}}} < C\sqrt{\bar{\rho}} \quad \forall \rho < \bar{\rho}.$$

By combining the Hölder inequality and the Sobolev embedding (in the same spirit as in the proof of proposition 2.1) and (2.9) we get:

$$(2.15) \quad \frac{1}{2} \rho > I_p^{\alpha,\beta}(\rho) = \lim_{n \rightarrow \infty} \mathcal{E}_p^{\alpha,\beta}(u_n) \geq \limsup_{n \rightarrow \infty} h_{p,\rho}(\|u_n\|_{H^{\frac{1}{2}}}) \quad \forall 0 < \rho < \rho_1$$

where $h_{p,\rho}(t) = \frac{1}{2}t^2 - C\rho^{3-p}t^{3p-6}$. Next notice that for a suitable $\rho_2 > 0$ we have

$$h_{p,\rho}(2\sqrt{\rho}) = 2\rho - C2^{3p-6}\rho^{\frac{p}{2}} > \rho \quad \forall 0 < \rho < \rho_2$$

and hence by (2.15) we get

$$\limsup_{n \rightarrow \infty} \|u_n\|_{H^{\frac{1}{2}}} \leq 2\sqrt{\rho} \quad \forall 0 < \rho < \min\{\rho_1, \rho_2\}.$$

Finally (2.14) implies trivially (2.11) □

Proposition 2.6. *For every $\alpha, \beta > 0$ there exists $\rho_1 = \rho_1(\alpha, \beta) > 0$ such that*

$$(2.16) \quad \frac{I_{\frac{8}{3}}^{\alpha,\beta}(\rho)}{\rho} < \frac{1}{2} \quad \forall 0 < \rho < \rho_1.$$

Moreover

$$(2.17) \quad \lim_{\rho \rightarrow 0} \frac{I_{\frac{8}{3}}^{\alpha,\beta}(\rho)}{\rho} = \frac{1}{2}.$$

Proof. The proof of (2.16) is identical to the proof of (2.9) (by choosing for instance $\gamma \in (\frac{3}{2}, 3)$).

Also the proof of (2.17) works as the proof of (2.10) provided that the estimate (2.15) is replaced by

$$(2.18) \quad \frac{1}{2} \rho > I_{\frac{8}{3}}^{\alpha,\beta}(\rho) = \lim_{n \rightarrow \infty} \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(u_n) \geq \limsup_{n \rightarrow \infty} h_{\frac{8}{3},\rho}(\|u_n\|_{H^{\frac{1}{2}}})$$

where $h_{\frac{8}{3},\rho}(t) = \frac{1}{2}t^2 - C\rho^{\frac{1}{3}}t^2 \geq \frac{1}{4}t^2$ (for ρ small enough) and $u_n \in S(\rho)$ is a minimizing sequence for $I_{\frac{8}{3}}^{\alpha,\beta}(\rho)$. In particular $h_{\frac{8}{3},\rho}(4\sqrt{\rho}) \geq 4\rho$ which by (2.18) implies

$$\limsup_{n \rightarrow \infty} \|u_n\|_{H^{\frac{1}{2}}} \leq C\sqrt{\rho}$$

for $0 < \rho < \rho_1$ with ρ_1 suitable small number. The proof can be concluded as the proof of (2.10). \square

3. THE CONCENTRATION-COMPACTNESS ARGUMENT

The main result of the section is the following proposition inspired by [12]. More precisely the question of compactness (up to translation) of minimizing sequences is reduced to the question of monotonicity of the function $\rho \rightarrow \rho^{-1}I_p^{\alpha,\beta}(\rho)$ (whose proof will be given in sections 4 and 5 respectively in the subcritical and critical case). Nevertheless it will be clear that some new ingredients, as well as some of the facts proved in previous sections, are needed along the proof of the next proposition.

Proposition 3.1. *Let $\alpha, \beta > 0$ and $p \in (2, \frac{8}{3}]$ be fixed. Let $\rho > 0$ be such that $I_p^{\alpha,\beta}(\rho) > -\infty$. Assume moreover*

$$(3.1) \quad \rho I_p^{\alpha,\beta}(\rho') < \rho' I_p^{\alpha,\beta}(\rho) \quad \forall 0 < \rho' < \rho.$$

Then for every minimizing sequence $u_n \in S(\rho)$ for $I_p^{\alpha,\beta}(\rho)$ there exists, up to subsequence, $\tau_n \in \mathbb{R}^3$ such that $u_n(\cdot + \tau_n)$ converge strongly to \bar{u} in $H^{\frac{1}{2}}$.

Lemma 3.1. *Let w_n be a sequence such that:*

$$\sup_n \|w_n\|_{H^{\frac{1}{2}}} < \infty$$

$$\|w_n\|_p \geq \epsilon_0 > 0 \text{ for a suitable } p \in (2, 3)$$

then there exists, up to subsequence, $\tau_n \in \mathbb{R}^3$ and a non trivial $\bar{w} \in H^{\frac{1}{2}}$ such that $w_n(\cdot + \tau_n)$ converges weakly to \bar{w} .

Proof. *First step:* $\forall q \in (2, 3) \exists \epsilon_q > 0$ s.t. $\|w_n\|_q > \epsilon_q$

If $q > p$ then by interpolation we get

$$\epsilon_0 \leq \|w_n\|_p \leq \|w_n\|_q^\theta \|w_n\|_2^{(1-\theta)}$$

where

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$$

and hence we can conclude since by assumption $\sup_n \|w_n\|_2 < \infty$.

If $q < p$ then by interpolation we get

$$\epsilon_0 \leq \|w_n\|_p \leq \|w_n\|_q^\theta \|w_n\|_3^{(1-\theta)}$$

where

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{3}$$

and we can conclude by the Sobolev embedding $H^{\frac{1}{2}} \subset L^3$ that in turn implies $\sup_n \|w_n\|_3 < \infty$.

Second step: if $v_n = (1 - \Delta)^{-\frac{1}{4}} w_n$ then $\|v_n\|_{\frac{10}{3}} \geq \delta > 0$ and $\sup_n \|v_n\|_{H^1} < \infty$. The estimate $\sup_n \|v_n\|_{H^1} < \infty$ follows from the identity

$$\|v_n\|_{H^1} = \|w_n\|_{H^{\frac{1}{2}}}.$$

By combining the inhomogeneous fractional Gagliardo-Nirenberg inequality (see [6] Prop. 4.2) with the first step we get:

$$0 < \epsilon_{\frac{5}{2}} \leq \|w_n\|_{\frac{5}{2}} = \|(1 - \Delta)^{\frac{1}{4}} v_n\|_{L^{\frac{5}{2}}} \leq C \|v_n\|_{H^1}^{\frac{1}{2}} \|v_n\|_{\frac{10}{3}}^{\frac{1}{2}}$$

which concluded the proof of the second step.

Finally due to the properties of the sequence v_n we can apply Lemma I.1 proved in [12] to deduce

$$\sup_{y \in \mathbb{R}^3} \int_{B(y,1)} |v_n|^2 dx \geq C > 0.$$

As a consequence, by the Rellich Theorem we deduce, up to subsequence, the existence of τ_n such that $v_n(\cdot + \tau_n)$ has a nontrivial weak limit \bar{v} different from zero. Eventually we conclude that also $w_n(\cdot + \tau_n) = (1 - \Delta)^{\frac{1}{4}} v_n(\cdot + \tau_n)$ converges weakly to $(1 - \Delta)^{\frac{1}{4}} \bar{v}$ which is different from zero. \square

Proof of Proposition 3.1.

Recall that by Proposition 2.1 (in the case $p \in (2, \frac{8}{3})$) and Proposition 2.2 (in the case $p = \frac{8}{3}$) we can assume $\sup_n \|u_n\|_{H^{\frac{1}{2}}} < \infty$.

First step: no-vanishing

First we prove the following

Claim $\exists \epsilon_0 > 0$ s.t. $\|u_n\|_p \geq \epsilon_0$

Assume it is not true then $\lim_{n \rightarrow \infty} \|u_n\|_p = 0$ and in particular

$$\begin{aligned} I_p^{\alpha, \beta}(\rho) &= \lim_{n \rightarrow \infty} \frac{1}{2} \|u_n\|_{H^{\frac{1}{2}}}^2 \\ &+ \alpha \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x - y|} dx dy - \beta \|u_n\|_p^p \geq \lim_{n \rightarrow \infty} \frac{1}{2} \|u_n\|_2^2 = \frac{1}{2} \rho \end{aligned}$$

which is in contradiction with (2.9) and (2.16).

By combining the claim with lemma 3.1 we get the existence, up to subsequence, of τ_n such that

$$v_n = u_n(\cdot + \tau_n)$$

has a weak limit \bar{v} different from zero.

Second step: v_n converges strongly to \bar{v} in $H^{\frac{1}{2}}$

It is sufficient to prove that $v_{n,\rho}$ converges strongly to \bar{v} in L^2 (then the strong convergence in $H^{\frac{1}{2}}$ follows by the fact that v_n is a minimizing sequence for $I_p^{\alpha,\beta}(\rho)$). In particular it is sufficient to prove that $\|\bar{v}\|_2^2 = \rho$. Assume by the absurd that $\|\bar{v}\|_2^2 = \delta \in (0, \rho)$, then since L^2 and $H^{\frac{1}{2}}$ are Hilbert spaces we get:

$$(3.2) \quad \|v_n - \bar{v}\|_2^2 = \rho - \delta + o(1)$$

and also

$$(3.3) \quad \|v_n - \bar{v}\|_{H^{\frac{1}{2}}}^2 = \|v_n\|_{H^{\frac{1}{2}}}^2 - \|\bar{v}\|_{H^{\frac{1}{2}}}^2 + o(1).$$

Moreover, up to subsequence, we can assume that

$$v_n(x) \rightarrow \bar{v}(x) \text{ a.e. } x \in \mathbb{R}^3.$$

Hence via the Brézis-Lieb Lemma (see [3]) we get

$$(3.4) \quad \|v_n - \bar{v}\|_p^p = \|v_n\|_p^p - \|\bar{v}\|_p^p + o(1)$$

and by [15]

$$(3.5) \quad \begin{aligned} & \int \int \frac{|(v_n - \bar{v})(x)|^2 |(v_n - \bar{v})(y)|^2}{|x - y|} dx dy \\ &= \int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x - y|} dx dy - \int \int \frac{|\bar{v}(x)|^2 |\bar{v}(y)|^2}{|x - y|} dx dy + o(1). \end{aligned}$$

By combining (3.2), (3.3), (3.4), (3.5) and the fact that v_n is a minimizing sequence for $I_p^{\alpha,\beta}(\rho)$ we get

$$\begin{aligned} I_p^{\alpha,\beta}(\rho) &= \mathcal{E}_p^{\alpha,\beta}(v_n) + o(1) = \mathcal{E}_p^{\alpha,\beta}(v_n - \bar{v}) + \mathcal{E}_p^{\alpha,\beta}(\bar{v}) + o(1) \\ &\geq I_p^{\alpha,\beta}(\rho - \delta + o(1)) + I_p^{\alpha,\beta}(\delta) + o(1) \end{aligned}$$

and in particular by the continuity of the function $I_p^{\alpha,\beta}(\rho)$ (see propositions 2.3 and 2.4) we get

$$(3.6) \quad I_p^{\alpha,\beta}(\rho) \geq I_p^{\alpha,\beta}(\rho - \delta) + I_p^{\alpha,\beta}(\delta).$$

Next notice that by (3.1) we get

$$I_p^{\alpha,\beta}(\rho - \delta) > \frac{\rho - \delta}{\rho} I_p^{\alpha,\beta}(\rho) \text{ and } I_p^{\alpha,\beta}(\delta) > \frac{\delta}{\rho} I_p^{\alpha,\beta}(\rho)$$

which imply

$$I_p^{\alpha,\beta}(\rho - \delta) + I_p^{\alpha,\beta}(\delta) > I_p^{\alpha,\beta}(\rho)$$

hence contradicting (3.6). \square

4. PROOF OF THEOREM 0.1

The proof of (0.10) follows from (1.2). Hence we shall focus on the proof of compactness (up to translation) of the minimizing sequences to $I_p^{\alpha,\beta}(\rho)$. By proposition 3.1 it is sufficient to prove that there exists $\rho = \rho(p) > 0$ such that

$$(4.1) \quad \mu \rightarrow \frac{I_p^{\alpha,\beta}(\mu)}{\mu} \text{ is decreasing in } (0, \rho).$$

Claim

Assume (4.1) is false then there exists a sequence $(\rho_n, u_n) \in (0, \infty) \times H^{\frac{1}{2}}$ such that:

$$(4.2) \quad \|u_n\|_2^2 = \rho_n > 0;$$

$$(4.3) \quad \mathcal{E}_p^{\alpha,\beta}(u_n) = I_p^{\alpha,\beta}(\rho_n);$$

$$(4.4) \quad 2\alpha \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy - \beta(p-2) \|u_n\|_p^p = 0;$$

$$(4.5) \quad \|u_n\|_{H^{\frac{1}{2}}} = o(1).$$

We first show how to conclude by assuming the claim.

First case: $2 < p < 12/5$

By combining the Hardy-Littlewood-Sobolev inequality with the interpolation inequality we get:

$$\int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy \leq C \|u_n\|_{12/5}^4 \leq C \|u_n\|_p^{\frac{p}{(3-p)}} \|u_n\|_3^{\frac{12-5p}{(3-p)}}.$$

Thanks to (4.4) and the Sobolev inequality we get:

$$\|u_n\|_p^p \leq C \|u_n\|_p^{\frac{p}{(3-p)}} \|u_n\|_{H^{\frac{1}{2}}}^{\frac{12-5p}{(3-p)}} = o(1) \|u_n\|_p^{\frac{p}{(3-p)}}$$

where we have used (4.5). Since $p < \frac{p}{(3-p)}$ and $\frac{12-5p}{(3-p)} > 0$ we get $\|u_n\|_p = 0$ for $n \geq \bar{n}$ which is absurd since $u_n \neq 0$ (see (4.2)).

Second case: $p = 12/5$

Due to (4.4) and Hardy-Littlewood-Sobolev inequality we get

$$\|u_n\|_{12/5}^{12/5} = \frac{5\alpha}{\beta} \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy \leq C \|u_n\|_{12/5}^4$$

which by (4.5) implies $u_n = 0$ for $n \geq \bar{n}$. Hence we get a contradiction with (4.2).

Third case: $12/5 < p < 8/3$

By combining (4.4) with interpolation inequality we get

$$\begin{aligned} \|u_n\|_p^p &= \frac{2\alpha}{\beta(p-2)} \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy \\ &\leq C \|u_n\|_{12/5}^4 \leq c \|u_n\|_2^{\frac{2(5p-12)}{3(p-2)}} \|u_n\|_p^{\frac{2p}{3(p-2)}} = o(1) \|u_n\|_p^{\frac{2p}{3(p-2)}} \end{aligned}$$

where we have used (4.5). Since we are assuming $p < 8/3$ we get $p < \frac{2p}{3(p-2)}$ and hence the estimate above implies $u_n = 0$ for $n > \bar{n}$. This is in contradiction with (4.2).

Proof of the Claim

Assume (4.1) it is not true then

$$\exists \delta_n \rightarrow 0 \text{ and } \gamma_n \in (0, \delta_n)$$

such that

$$(4.6) \quad \frac{I_p^{\alpha,\beta}(\delta_n)}{\delta_n} \geq \frac{I_p^{\alpha,\beta}(\gamma_n)}{\gamma_n}.$$

We also introduce $y_n = \min_{[0, \delta_n]} \frac{I_p^{\alpha,\beta}(\rho)}{\rho}$. Due to (2.10) we get

$$(4.7) \quad y_n < \frac{1}{2}.$$

Moreover due to proposition 2.3 and (2.10) we can define

$$\rho_n = \min \left\{ \rho \leq \delta_n \text{ s.t. } \frac{I_p^{\alpha,\beta}(\rho)}{\rho} = y_n \right\}$$

which satisfy:

$$(4.8) \quad 0 < \rho_n < \delta_n$$

(use (4.6) for the upper bound, (4.7) and (2.9) for the lower bound);

$$(4.9) \quad \rho_n \rightarrow 0;$$

$$(4.10) \quad \frac{I_p^{\alpha,\beta}(\rho_n)}{\rho_n} < \frac{I_p^{\alpha,\beta}(\rho)}{\rho} \quad \forall \rho < \rho_n.$$

By combining proposition 3.1 with (4.10) we deduce that

$$(4.11) \quad \exists u_n \in S(\rho_n) \text{ s.t. } \mathcal{E}_p^{\alpha,\beta}(u_n) = I_p^{\alpha,\beta}(\rho_n).$$

Moreover by definition of ρ_n and (4.8) we deduce that

$$(4.12) \quad (0, \delta_n) \ni \rho \rightarrow \frac{I_p^{\alpha,\beta}(\rho)}{\rho} \text{ has a global minimum in } \rho_n \in (0, \delta_n).$$

As a consequence of the above facts we can deduce that u_n is a local minimum of the functional

$$H^{\frac{1}{2}} \setminus \{0\} \ni u \rightarrow \frac{\mathcal{E}_p^{\alpha,\beta}(u)}{\|u\|_2^2}$$

(indeed if it is not true

$$\exists u_{k,n} \rightarrow u_n \text{ in } H^{\frac{1}{2}} \text{ s.t. } \frac{\mathcal{E}_p^{\alpha,\beta}(u_{k,n})}{\|u_{k,n}\|_2^2} < \frac{\mathcal{E}_p^{\alpha,\beta}(u_n)}{\|u_n\|_2^2} = \frac{I_p^{\alpha,\beta}(\rho_n)}{\rho_n}$$

and hence $\frac{I_p^{\alpha,\beta}(\|u_{k,n}\|_2^2)}{\|u_{k,n}\|_2^2} < \frac{I_p^{\alpha,\beta}(\rho_n)}{\rho_n}$; since $\lim_{k \rightarrow \infty} \|u_{k,n}\|_2^2 = \|u_n\|_2^2 = \rho_n$ we get a contradiction with (4.12)).

Hence we can apply Proposition 1.1 to deduce that u_n satisfies the functional identity (1.1) and hence (4.4).

Finally to prove (4.5) recall that by looking at (2.11) one can show that indeed $\|u_n\|_{H^{\frac{1}{2}}} \leq C\sqrt{\rho_n}$.

□

5. PROOF OF THEOREM 0.3

The proof of (0.12) follows from (1.2). Hence we shall focus on the proof of compactness (up to translation) of the minimizing sequences to $I_{\frac{8}{3}}^{\alpha,\beta}(\rho)$.

Due to proposition 3.1 it is sufficient to prove that there exists $\bar{\rho} > 0$ small enough such that

$$(5.1) \quad \mu \rightarrow \frac{I_{\frac{8}{3}}^{\alpha,\beta}(\mu)}{\mu} \text{ is decreasing in } (0, \bar{\rho}).$$

In the same spirit of the proof of theorem 0.1 we can prove the following

Claim

Assume (5.1) is false then there exists a sequence $(\rho_n, u_n) \in (0, \infty) \times H^{\frac{1}{2}}$ such that:

$$(5.2) \quad \|u_n\|_2^2 = \rho_n > 0;$$

$$(5.3) \quad \lim_{n \rightarrow \infty} \|u_n\|_2 = 0;$$

$$(5.4) \quad \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(u_n) = I_{\frac{8}{3}}^{\alpha,\beta}(\rho_n);$$

$$(5.5) \quad 2\alpha \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy - \frac{2}{3}\beta \|u_n\|_{\frac{8}{3}}^{\frac{8}{3}} = 0.$$

The proof of the claim is identical to the proof of the analogue claim used along the proof of theorem 0.1. Once the claim is established then we combine (5.5), the Hardy-Littlewood-Sobolev inequality and the interpolation inequality to get

$$\|u_n\|_{\frac{8}{3}}^{\frac{8}{3}} = 3\frac{\alpha}{\beta} \int \int \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dx dy \leq C \|u_n\|_{\frac{12}{5}}^4 \leq C \|u_n\|_{\frac{4}{3}}^{\frac{4}{3}} \|u_n\|_{\frac{8}{3}}^{\frac{8}{3}}.$$

By using (5.3) we deduce $\|u_n\|_{\frac{8}{3}} = 0$ for any n large enough, which is in contradiction with (5.2).

6. PROOF OF THEOREM 0.2

The proof of the theorem 0.2 follows by combining the next two propositions. In the sequel the energy $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}$ is the one defined in (0.13).

Proposition 6.1. *The following facts are equivalent:*

- $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) = -\infty$
- $\exists \varphi \in S(\rho)$ s.t. $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) < 0$.

Proof. If $I_{\frac{8}{3}}^{\alpha,\beta}(\rho) = -\infty$ then there exists $\varphi \in S(\rho)$ such that $\mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) < 0$ and hence $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) \leq \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) < 0$.

Next we prove the opposite implication. We introduce $\varphi_\theta(x) = \theta^{\frac{3}{2}}\varphi(\theta x)$ then by a scaling argument

$$\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = \theta \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi).$$

Next notice that

$$\begin{aligned} \|\varphi_\theta\|_{H^{\frac{1}{2}}}^2 - \|\varphi_\theta\|_{\dot{H}^{\frac{1}{2}}}^2 &= \int \sqrt{1+|\xi|^2} \left| \hat{\varphi}\left(\frac{\xi}{\theta}\right) \right|^2 \frac{d\xi}{\theta^3} - \theta \int |\xi| |\hat{\varphi}(\xi)|^2 d\xi \\ &= \int (\sqrt{1+\theta^2|\xi|^2} - \theta|\xi|) |\hat{\varphi}(\xi)|^2 d\xi = \int \frac{1}{\sqrt{1+\theta^2|\xi|^2} + \theta|\xi|} |\hat{\varphi}(\xi)|^2 d\xi = o(1) \text{ as } \theta \rightarrow \infty. \end{aligned}$$

Finally we get

$$\begin{aligned} \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) &= \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) + \frac{1}{2}(\|\varphi_\theta\|_{H^{\frac{1}{2}}}^2 - \|\varphi_\theta\|_{\dot{H}^{\frac{1}{2}}}^2) \\ &= \theta \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) + o(1) \end{aligned}$$

and hence

$$I_{\frac{8}{3}}^{\alpha,\beta}(\rho) \leq \lim_{\theta \rightarrow \infty} \mathcal{E}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = -\infty.$$

□

Proposition 6.2. *The following facts are equivalent:*

- $\tilde{\mathcal{E}}_p^{\alpha,\beta}(\varphi) \geq 0 \ \forall \varphi \in H^{\frac{1}{2}};$
- $\left(\frac{27\alpha}{\beta^3}\right)^{\frac{1}{8}} \geq \sqrt{2}S.$

Proof. Let $\varphi_\theta(x) = \varphi\left(\frac{x}{\theta}\right)$ then we have

$$\tilde{\mathcal{E}}_p^{\alpha,\beta}(\varphi) \geq 0 \ \forall \varphi \in H^{\frac{1}{2}}$$

if and only if

$$\tilde{\mathcal{E}}_p^{\alpha,\beta}(\varphi_\theta) \geq 0 \ \forall \varphi \in H^{\frac{1}{2}}, \theta \in (0, \infty).$$

By explicit computation this is equivalent to

$$\begin{aligned} \frac{1}{2}\theta^2\|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 + \alpha\theta^5 \int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy - \beta\theta^3\|\varphi\|_{\frac{8}{3}}^{\frac{8}{3}} &\geq 0 \\ \forall \varphi \in H^{\frac{1}{2}}, \theta \in (0, \infty). \end{aligned}$$

Hence the condition $\tilde{\mathcal{E}}_p^{\alpha,\beta}(\varphi) \geq 0 \ \forall \varphi \in H^{\frac{1}{2}}$ can be rewritten as follows:

$$(6.1) \quad \inf_{\theta \in (0, \infty)} \psi_\varphi^{\alpha,\beta}(\theta) \geq 0 \ \forall \varphi \in H^{\frac{1}{2}}, \theta \in (0, \infty)$$

where

$$\psi_\varphi^{\alpha,\beta}(\theta) = \frac{1}{2}\|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 + \alpha\theta^3 \int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy - \beta\theta\|\varphi\|_{\frac{8}{3}}^{\frac{8}{3}} \geq 0.$$

By elementary computation we get

$$\begin{aligned} \inf_{(0, \infty)} \psi_\varphi^{\alpha,\beta}(\theta) &= \psi_\varphi^{\alpha,\beta} \left(\|\varphi\|_{\frac{8}{3}}^{\frac{4}{3}} \sqrt{\frac{\beta(\int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy)^{-1}}{3\alpha}} \right) \\ &= \frac{1}{2}\|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 + \left(\alpha \left(\frac{\beta}{3\alpha} \right)^{\frac{3}{2}} - \beta \sqrt{\frac{\beta}{3\alpha}} \right) \frac{\|\varphi\|_{\frac{8}{3}}^{\frac{4}{3}}}{\sqrt{\int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy}} \\ &= \frac{1}{2}\|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 - \frac{2}{3}\beta \sqrt{\frac{\beta}{3\alpha}} \frac{\|\varphi\|_{\frac{8}{3}}^{\frac{4}{3}}}{\sqrt{\int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy}}. \end{aligned}$$

Hence the condition (6.1) becomes

$$4\sqrt{\frac{\beta^3}{27\alpha}}\|\varphi\|_{\frac{8}{3}}^{\frac{4}{3}} \leq \|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 \sqrt{\int \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dx dy} \ \forall \varphi \in \dot{H}^{\frac{1}{2}}$$

and we can conclude since by definition S is the best constant in the inequality

$$\|\varphi\|_{\frac{8}{3}} \leq S \|\varphi\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \left(\int \int \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy \right)^{\frac{1}{8}} \quad \forall \varphi \in \dot{H}^{\frac{1}{2}}.$$

□

7. PROOF OF THEOREM 0.4

We shall need the following lemma.

Lemma 7.1. *The following dichotomy happens:*

$$(7.1) \quad \text{either } \tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) = 0 \text{ or } I_{\frac{8}{3}}^{\alpha,\beta}(\rho) = -\infty.$$

Moreover there exists $\tilde{\rho} > 0$ such that

$$(7.2) \quad \tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) = 0 \quad \forall \rho \in (0, \tilde{\rho})$$

Proof. *First step:* $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) \leq 0$

We fix $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\|\varphi\|_2^2 = \rho$ and $\varphi_\theta = \theta^{\frac{3}{2}} \varphi(\theta x)$. Then $\|\varphi\|_2^2 = \rho$. By direct computation

$$\begin{aligned} \bullet \quad & \|\varphi_\theta\|_{\frac{8}{3}}^{\frac{8}{3}} = \theta \|\varphi\|_{\frac{8}{3}}^{\frac{8}{3}} \\ \bullet \quad & \int \int \frac{|\varphi_\theta(x)|^2 |\varphi_\theta(y)|^2}{|x-y|} dx dy = \theta \int \int \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy. \\ \bullet \quad & \|\varphi_\theta\|_{\dot{H}^{\frac{1}{2}}}^2 = \theta \|\varphi\|_{\dot{H}^{\frac{1}{2}}}^2 \end{aligned}$$

In particular we get

$$\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = \theta \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi)$$

which implies

$$\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) \leq \lim_{\theta \rightarrow 0} \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = 0.$$

Second step: if $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) < 0$ then $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) = -\infty$

Let $\varphi \in S(\rho)$ be such that $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) < 0$ then arguing as above we get

$$\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = \theta \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi)$$

where $\varphi_\theta = \theta^{\frac{3}{2}} \varphi(\theta x)$. Hence

$$\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) \leq \lim_{\theta \rightarrow \infty} \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi_\theta) = \lim_{\theta \rightarrow \infty} \theta \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(\varphi) = -\infty.$$

The proof of (7.1) follows easily.

Next we focus on (7.2). Notice that by combining Hölder inequality with the Sobolev embedding $\dot{H}^{\frac{1}{2}} \subset L^3$ we get

$$(7.3) \quad \tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(u) \geq \frac{1}{2}\|u\|_{\dot{H}^{\frac{1}{2}}} - \beta\|u\|_3^2\|u\|_2^{\frac{2}{3}} \geq \frac{1}{2}\|u\|_{\dot{H}^{\frac{1}{2}}} - C\|u\|_{\dot{H}^{\frac{1}{2}}}^2\rho^{\frac{1}{3}} \quad \forall u \in S(\rho).$$

In particular if ρ is small then $\tilde{\mathcal{E}}_{\frac{8}{3}}^{\alpha,\beta}(u) \geq 0$ for any $u \in S(\rho)$ and hence

$$\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho) \geq 0.$$

By combining this fact with (7.1) we deduce (7.2). □

Proof of theorem 0.4. Let $\rho_* > 0$ be such that

$$\frac{1}{2} - C\rho_*^{\frac{1}{3}} > 0$$

where C is the universal constant that appears in (7.3). Let $\tilde{\rho}$ be as in lemma 7.1. Then by using lemma 7.1 $\tilde{I}_{\frac{8}{3}}(\rho) = 0$ for every $\rho < \min\{\tilde{\rho}, \rho_*\}$. By combining this fact with (7.3) we deduce that if u_n is a minimizing sequence for $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho)$ with $\rho < \min\{\tilde{\rho}, \rho_*\}$ then

$$\lim_{n \rightarrow \infty} \|u_n\|_{\dot{H}^{\frac{1}{2}}} = 0$$

In particular it implies that if $v \in S(\rho)$ is a minimizer for $\tilde{I}_{\frac{8}{3}}^{\alpha,\beta}(\rho)$ with $\rho < \min\{\tilde{\rho}, \rho_*\}$ then $v = 0$ (which is absurd since if $v \in S(\rho)$ for $\rho > 0$ then $v \neq 0$). □

8. APPENDIX

This section is devoted to the proof of the inequality (0.11), whose best constant is involved in the statement of theorem 0.2.

Proposition 8.1. *There exists $C > 0$ such that*

$$\|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)} \leq C\|\varphi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{\frac{1}{2}} \left(\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy \right)^{\frac{1}{8}}$$

Proof. By using basic facts on Fourier transform the previous estimate is equivalent to the following one:

$$(8.1) \quad \|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)} \leq C\|\varphi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{\frac{1}{2}} \|\varphi\|_{\dot{H}^{-1}(\mathbb{R}^3)}^{\frac{1}{4}}.$$

Notice that we have the following Gagliardo-Nirenberg inequality

$$\| |D| \varphi \|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq C \|\varphi\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \| |D|^{\frac{3}{2}} \varphi \|_{L^{\frac{8}{7}}(\mathbb{R}^3)}^{\frac{2}{3}}$$

that can be rewritten as follows:

$$\|\varphi\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq C \| |D|^{-1} \varphi \|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \| |D|^{\frac{1}{2}} \varphi \|_{L^{\frac{8}{7}}(\mathbb{R}^3)}^{\frac{2}{3}}.$$

Next we replace φ by $|\varphi|^2$ and we get

$$\|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^2 \leq C \| |D|^{-1} |\varphi|^2 \|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \| |D|^{\frac{1}{2}} |\varphi|^2 \|_{L^{\frac{8}{7}}(\mathbb{R}^3)}^{\frac{2}{3}}$$

that in turn by the fractional chain rule implies

$$\|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^2 \leq C \| |D|^{-1} |\varphi|^2 \|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \| |D|^{\frac{1}{2}} \varphi \|_{L^2(\mathbb{R}^3)}^{\frac{2}{3}} \|\varphi\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^{\frac{2}{3}}.$$

The last inequality is equivalent to (8.1). □

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